INTERIOR L^p - ESTIMATES FOR ELLIPTIC AND PARABOLIC SCHRÖDINGER TYPE OPERATORS AND LOCAL A_p -WEIGHTS

ISOLDA CARDOSO – PABLO VIOLA – BEATRIZ VIVIANI

Abstract. Let Ω be a non-empty open proper and connected subset of $\mathbb{R}^n$. Consider the elliptic Schrödinger type operator $L_E u = A_E u + Vu = -\Sigma_{ij} a_{ij}(x)$ $u_{x_i x_j} + Vu$ in Ω, and the linear parabolic operator $L_P u = A_P u + Vu = u_t - \Sigma a_{ij}(x,t) u_{x_i x_j} + Vu$ in $\Omega_T = \Omega \times (0,T)$, where the coefficients $a_{ij} \in VMO$ and the potential V satisfies a reverse-Hölder condition. The aim of this paper is to obtain a priori estimates for the operators L_E and L_P in weighted Sobolev spaces involving the distance to the boundary and weights in a local-A_p class.

Mathematics Subject Classification (2010): Primary: 35J10; Secondary: 35B45; 42B35.

1. Introduction

Let Ω be a non-empty open proper and connected subset of $\mathbb{R}^n$. We are going to consider the following two operators: the elliptic Schrödinger type operator

$$L_E u = A_E u + Vu = -\sum_{ij} a_{ij}(x) u_{x_i x_j} + Vu$$

in Ω, and the linear parabolic operator

$$L_P u = A_P u + Vu = u_t - \sum a_{ij}(x',t) u_{x_i x_j} + Vu$$

in $\Omega_T = \Omega \times (0,T)$, with $T > 0$, under the following assumptions:

(1) $a_{ij} = a_{ji}$, and
$$\frac{1}{C}|\xi|^2 \leq \sum_{ij} a_{ij}(.)\xi_i \bar{\xi_j} \leq C|\xi|^2$$
for a.e. $x \in \Omega$ or $x = (x',t) \in \Omega_T$, respectively;

(2) $a_{ij} \in L^\infty \cap VMO(\mathbb{R}^n)$. Here we have the space of functions of vanishing mean oscillation defined as
$$VMO(\mathbb{R}^n) = \left\{ g \in BMO(\mathbb{R}^n) : \eta(r) \to 0, r \to 0^+ \right\},$$
where
$$\eta(r) = \sup_{\rho \leq r} \sup_{x \in \mathbb{R}^n} \left(\frac{1}{|B_\rho(x)|} \int_{B_\rho(x)} |g(y) - g_{B_\rho}| dy \right).$$
Here $g_{B_\rho} = |B(\rho(x))|^{-1} \int_{B_\rho(x)} g(y)\, dy$. The parabolic $VMO(\mathbb{R}^{n+1})$ is defined in the same way, except this time we take the supremum over the parabolic balls (see section 2.1.1);

(3) The potential $V \geq 0$ satisfies a reverse Hölder condition of order q, shortly $V \in RH_q$, which means that

$$\Big(\frac{1}{|B|}\int_B V^q dx\Big)^{1/q} \leq \frac{1}{|B|}\int_B V dx, \tag{1.1}$$

where the ball B is in $\mathbb{R}^n$.

Sometimes we will use A for either the operators A_E or A_P, and Λ for either the subset Ω or Ω_T.

When the coefficients a_{ij} are at least uniformly continuous, existence and uniqueness results together with a-priori $W^{2,p}$ estimates are well known (see e.g. [7]). The theory for operators with discontinuous coefficients, in the sense of VMO, goes back to the 90's with the works of Chiarenza-Frasca-Longo in [4] and [5] for elliptic operators and Bramanti-Cerutti in [3] for the parabolic case. Since then, many authors have considered this problem in different situations and contexts. The Schrödinger operator when A is the Laplacian and the potential V satisfies the reverse-Hölder condition (3), was studied by Shen in [15] and related results when $V(x) = |x|$ (Hermite operator) have been proved by Thangavelu in [16]. For the elliptic type Schrödinger operator under consideration, a global $W^{2,p}(\mathbb{R}^n)$ estimate and the existence and uniqueness results deduced from them were obtained in [2]. We are interested in obtaining a priori interior estimates in weighted Sobolev spaces for the operator L, where L is either the elliptic Schrödinger type operator L_E or the parabolic operator L_P, defined in a non necessarily bounded domain. We follow the strategy adopted in [2]. First we get a weighted version of the a priori estimates obtained in [4] and in [3] for the principal operator A_E and A_P respectively. Thanks to these estimates we are reduced to prove a weighted L^p bound on Vu in terms on Lu. Then, we give a representation formula for Vu by means of the fundamental solution of a constant coefficient operator of the type $A_0 + V$, for which a global estimate was proved by Dziubanski in [6] for L_E and by Kurata in [11] for L_P. These representation formulas involve suitable integral operators with positive kernel, applied to Lu, and their positive conmutators, applied to the second order derivatives of u.

In order to prove that these operators are bounded on weighted L^p, we use local maximal functions, $M_{\text{loc}}f$ (see section 2), defined in a proper open set imbedded in a metric space. This maximal operator and the classes of weight involved $A_{p,\text{loc}}$ (see below), were first studied by Nowak and Stempak in [12] when $\Omega = (0,\infty)$ and by Lin and Stempak in [9] for $\Omega = \mathbb{R}^n \setminus \{0\}$. In a general setting, that is in metric spaces, this maximal operator and the corresponding classes of weights were considered by Harboure, Salinas and Viviani in [8] and by Lin, Stempak and Wan in [10].

We consider the local weights class $A_{p,\text{loc}}$ defined as follows: let (X, d) be a metric space and let Λ be a nonempty open proper subset of X, if $0 < \beta < 1$ we define the family of balls

$$F_\beta = \big\{B = B(x_B, r_B) : x_B \in \Gamma, r_B < \beta d(x_B, \Lambda^C)\big\},$$

where $d(x_B, \Lambda^C)$ denotes the distance from the center x_B of the ball B to the complementary set of Λ. Given a Borel measure μ defined on Λ, for $1 < p < \infty$,

we define

$$w \in A^{\beta}_{p,\mathrm{loc}}(\Lambda) \ \text{ iff } \ \sup_{B\in\mathcal{F}_\beta} \frac{1}{\mu(B)}\Big(\int_B w d\mu\Big)^{1/p}\Big(\int_B w^{-p/p'}d\mu\Big)^{1/p'} < \infty. \tag{1.2}$$

We remark that the classes $A^{\beta}_{p,\mathrm{loc}}(\Lambda)$ are independent of β, as was shown in [8]. In view of this fact, we shall refer to theses weights as $A_{p,\mathrm{loc}}(\Lambda)$. We also consider the following weighted Sobolev spaces, defined in $\mathbb{R}^n$ and $\mathbb{R}^{n+1}$, respectively:

$$W^{2,p}_{\delta,w}(\Omega) = \Big\{u \in L^1_{\mathrm{loc}}(\Omega) : \|u\|_{W^{2,p}_{\delta,w}(\Omega)} = \sum_{|\gamma|\le 2} \|\delta^{|\gamma|}D^\gamma u\|_{L^p_w(\Omega)} < \infty\Big\},$$

and

$$W^{2,p}_{\delta,w}(\Omega_T)=\Big\{u\in L^1_{\mathrm{loc}}(\Omega_T):\|u\|_{W^{2,p}_{\delta,w}(\Omega_T)}=\sum_{|\gamma|\le 2}\|\delta^{|\gamma|}D^\gamma_x u\|_{L^p_w(\Omega_T)}+\|\delta^2 D_t u\|_{L^p_w(\Omega_T)}<\infty\Big\},$$

where $\delta(x) = \min\{1, d(x,\Lambda^C)\}$, with either $\Lambda = \Omega$ or Ω_T, and d denotes the corresponding distance.

We will prove the following results:

Theorem 1.1. *Let Ω be a nonempty, proper, open and connected subset of $\mathbb{R}^n$. Let $p \in (1, q]$ and $w \in A_{p,loc}(\Omega)$. If $u \in W^{2,p}_{\delta,w}(\Omega)$ is a solution of*

$$Lu = Au + Vu = -\sum_{i,j} a_{ij}u_{x_i x_j} + Vu = f \qquad \text{in } \Omega,$$

under the assumptions (1), (2) and (3), then

$$\|u\|_{W^{2,p}_{\delta,w}(\Omega)} + \|\delta^2 V u\|_{L^p_w(\Omega)} \le C\big[\|\delta^2 f\|_{L^p_w(\Omega)} + \|u\|_{L^p_w(\Omega)}\big],$$

where $\delta(x) = \min\{1, d(x,\Omega^C)\}$, $x \in \mathbb{R}^n$.

The parabolic version of this theorem goes as follows:

Theorem 1.2. *Let Ω be a nonempty, proper, open and connected subset of $\mathbb{R}^n$. For $T > 0$ define $\Omega_T = \Omega \times \big(0, T\big)$. Let $p \in (1, q]$ and $w \in A_{p,loc}(\Omega_T)$. If $u \in W^{2,p}_{\delta,w}(\Omega_T)$ is a solution of*

$$Lu = Au + Vu = u_t - \sum_{i,j} a_{ij}u_{x_i x_j} + Vu = f \qquad \text{in } \Omega_T,$$

under the assumptions (1), (2) and (3), then

$$\|u\|_{W^{2,p}_{\delta,w}(\Omega_T)} + \|\delta^2 V u\|_{L^p_w(\Omega_T)} \le C\big[\|\delta^2 f\|_{L^p_w(\Omega_T)} + \|u\|_{L^p_w(\Omega_T)}\big],$$

where $\delta(x',t) = \min\{1, d((x',t),\Omega_T^C)\}$.

We note that, as it is easy to check, $w(x) = \delta^\alpha(x)$ belongs to $A_{p,\mathrm{loc}}$ for any exponent $\alpha \in \mathbb{R}$. Therefore the data function f appearing on the right hand side of Theorem 1.1 and Theorem 1.2 could increase polynomially when approaching the boundary of Ω or Ω_T and still we might have some control for the derivatives of the solution up to the order 2.

The paper is organized as follows: in Section 2 we put together the preliminary definitions and results, and prove some useful lemmas; in Section 3 we prove some results that will build the proof of the Main Theorem for the operator L_E, and in Section 4 we show similar results for the operator L_P. Finally, in Section 5 we end up proving the main results stated above: Theorems 1.1 and 1.2.

Acknowledgments. The first author is partially supported by Universidad Nacional de Rosario and a grant from Consejo Nacional de Investigaciones Científicas y Técnicas (CONICET). The second author is partially supported by Núcleo Consolidado de Matemática Pura y Aplicada, Universidad Nacional del Centro de la Provincia de Buenos Aires and by CONICET. The third author is partially supported by grants from CONICET and Universidad Nacional del Litoral.

2. Preliminaries

2.1. **Definition and notations.**

2.1.1. *The parabolic setting.* The parabolic setting we are considering consists of $\mathbb{R}^{n+1}$ endowed with the following parabolic metric

$$d(x,y) = (|x'-y'|^2 + |t-s|)^{\frac{1}{2}},$$

where we write $x=(x',t), y=(y',s) \in \mathbb{R}^{n+1}$, with $x', y' \in \mathbb{R}^n$ and $t,s \in \mathbb{R}^+$. We denote the parabolic balls as usual:

$$B(x,r) = \{y \in \mathbb{R}^{n+1} : d(x,y) < r\}.$$

and its Lebesgue measure by $|B(x,r)| = c_n r^{n+2}$.

2.1.2. *The local maximal operator.* In this subsection we will denote by X a metric space satisfying the weak homogeneity property, that is, there exists a fix number N such that for any ball $B(x,r)$ there are no more than N points in the ball whose distance from each other is greater than $r/2$. Also Λ will mean any open proper and non empty subset of X such that all balls contained in Λ are connected sets and μ will be a Borel measure defined on Λ which satisfies a doubling condition on F_β, that is, there is some constant C_β such that for any ball $B \in F_\beta$

$$\mu(B) \leq C_\beta \mu(\tfrac{1}{2}B)$$

with $0 < \mu(B) < \infty$ for any ball $B \in \mathcal{F} = \bigcup_{0<\alpha<1} \mathcal{F}_\alpha$.

We shall use the following local maximal operator associated to $\mathcal{F}_\beta$: given $0 < \beta < 1$ and μ as above

$$M_{\mu,\beta} f(x) = \sup_{x \in B \in \mathcal{F}_\beta} \frac{1}{\mu(B)} \int_B |f| \, d\mu \tag{2.1}$$

for any $f \in L^1_{\mathrm{loc}}(\Lambda, d\mu)$ and $x \in \Lambda$. When μ is the Lebesgue measure we denote $M_{\mu,\beta} f$ by $M_{\beta,\mathrm{loc}} f$.

The boundness property for $M_{\mu,\beta} f$ is contained in the next Theorem:

Theorem 2.1 ([8], Theorem 1.1)**.** *Let X and Λ as above. Let $0 < \beta < 1$ and μ a Borel measure satisfying the doubling property on $\mathcal{F}_\beta$. Then, for $1 < p < \infty$, $M_{\mu,\beta} f$ is bounded on $L^p_w(\Lambda, w d\mu)$ if and only if $w \in A^\beta_{p,loc}(\Lambda)$.*

2.1.3. *The properties of the potential V.* The potential V satisfies assumption (3) and, as it is remarked in [2], the condition $V \in RH_q$ implies that for some $\epsilon > 0$ we have also that $V \in RH_{q+\epsilon}$, where the $RH_{q+\epsilon}$ constant of V is controlled in terms of the RH_q constant of V. They also remark the useful fact that the measure $V(y)dy$ is doubling.

Associated to the function $V \in RH_q$ there is a function $\rho(x)$, called *critical radious*, defined by Shen in [15]:

$$\rho(x) = \sup\left\{ r > 0 : \frac{r^2}{|B(x,r)|} \int_{B(x,r)} V(y)dy \leq 1 \right\}, \tag{2.2}$$

which, under our assumptions on V, is finite almost everywhere. We note that by definition of ρ, we have that

$$\frac{1}{\rho(x)^{n-2}} \int_{B(x,\rho(x))} V(y)dy \leq 1. \tag{2.3}$$

Shen also proved that the following inequalities hold:

$$C\Big(1 + \frac{|x-y|}{\rho(y)}\Big)^{\frac{1}{k_0}} \leq 1 + \frac{|x-y|}{\rho(x)} \leq C\Big(1 + \frac{|x-y|}{\rho(y)}\Big)^{\frac{1}{k_0}}, \tag{2.4}$$

for some $k_0 \in \mathbb{N}$ and any $x, y \in \mathbb{R}^n$ and

$$\frac{1}{r^n} \int_{B(x,r)} V(y)dy \leq C\Big(\frac{R}{r}\Big)^{\frac{n}{q}} \frac{1}{R^n} \int_{B(x,R)} V(y), \tag{2.5}$$

for any $0 < r < R < \infty$.

2.1.4. *Bounds for the fundamental solutions of the constant coefficient operators* L_0. Let us now consider the operator A, which denotes either A_E or A_P. For fixed $x_0 \in \Lambda$, where Λ denotes Ω or Ω_T, respectively, freeze the coefficients $a_{ij}(x_0)$ and denote L_0 the operator L with these constant coefficients.

Dziubanski in [6] proved that the elliptic operator L_0 has a fundamental solution $\Gamma(x_0; x, y)$ which satisfies that for any $k \in \mathbb{N}$ there exists a constant c_k (independent of x_0) such that

$$\Gamma(x_0; x, y) \leq c_k \frac{1}{\big(1 + \frac{|x-y|}{\rho(x)}\big)^k} \frac{1}{|x-y|^{n-2}}, \tag{2.6}$$

for any $x, y \in \mathbb{R}^n$, $x \neq y$. Here ρ is the critical radious associated to V defined in 2.2. We remark that the kernel

$$W(x,y) = V(y) \frac{1}{\big(1 + \frac{|x-y|}{\rho(x)}\big)^k} \frac{1}{|x-y|^{n-2}},$$

satisfies *Hörmander's condition of order* q, briefly *condition* $H_1(q)$, in the first variable (see Proposition 12 in [2]). This means that there exists a constant $C > 0$ such that for any $r > 0$ and any $x, x_0 \in \mathbb{R}^n$ with $|x - x_0| < r$, the following inequality holds:

$$\sum_{j=1}^{\infty} j|B(x_0, 2^j r)|^{\frac{1}{q'}} \Big(\int_{2^j r \leq |x_0 - y| \leq 2^{j+1} r} |W(x,y) - W(x_0,y)|^q dy \Big)^{\frac{1}{q}} \leq C. \tag{2.7}$$

Also, observe that from inequalities 2.4 we can replace $\rho(y)$ with $\rho(x)$ in the kernel W, possibly changing the integer k.

For the parabolic operator L_0, Kurata showed in Corollary 1 of [11] that it has a fundamental solution $\Gamma(x_0; x, y)$ which satisfies that for each $k \in \mathbb{N}$ there exists constants c_k and c_0 (independents of x_0) such that

$$\Gamma(x_0;x,y) \leq c_k \frac{1}{\left(1+\frac{d(x,y)}{\rho(x')}\right)^k} \frac{1}{|t-s|^{n/2}} e^{-c_0 \frac{|x'-y'|^2}{|t-s|}},$$

where d is the parabolic distance given in 2.1.1. Thus,

$$\Gamma(x_0;x,y) \leq c_k \frac{1}{\left(1+\frac{d(x,y)}{\rho(x')}\right)^k} \frac{1}{d(x,y)^n}. \tag{2.8}$$

The parabolic kernel, appearing on the right hand side of 2.8, also satisfies condition $H_1(q)$, as we prove in the next subsection.

2.2. **Previous Lemmas.**

Lemma 2.2. *The kernel*

$$W(x,y) = V(y') \frac{1}{\left(1+\frac{d(x,y)}{\rho(y')}\right)^k} \frac{1}{d(x,y)^n}$$

satisfies condition $H_1(q)$ for k large enough, that is, there exists a constant $C>0$ such that for every $r>0$, $x, x_0 \in \mathbb{R}^{n+1}$ with $d(x,x_0) < r$,

$$\sum_{j=1}^{\infty} j (2^j r)^{\frac{n+2}{q'}} \Big(\int_{2^j r < d(x_0,y) \leq 2^{j+1} r} |W(x,y) - W(x_0,y)|^q dy \Big)^{\frac{1}{q}} \leq C.$$

Proof. We follow the lines of Proposition 12 of [2]. As usual, we may assume $q > \frac{n}{2}$. Let $x, x_0, y \in \Omega_T$ be such that $d(x,x_0) \leq r$ and $d(y,x_0) \geq 2r$, so that in particular $d(x_0,y) \simeq d(x,y)$.

The first step is to compute

$$|W(x,y) - W(x_0,y)| \leq V(y') \Bigg(\frac{1}{\left(1+\frac{d(x_0,y)}{\rho(y')}\right)^k} \left| \frac{1}{d(x,y)^n} - \frac{1}{d(x_0,y)^n} \right| +$$
$$+ \frac{1}{d(x,y)^n} \left| \frac{1}{\left(1+\frac{d(x,y)}{\rho(y')}\right)^k} - \frac{1}{\left(1+\frac{d(x_0,y)}{\rho(y')}\right)^k} \right| \Bigg) = A + B.$$

We note that by the mean value Theorem

$$\left| \frac{1}{d(x,y)^n} - \frac{1}{d(x_0,y)^n} \right| \leq C \frac{d(x,x_0)}{d(x_0,y)^{n+1}},$$

Also

$$\left| \frac{1}{\left(1+\frac{d(x,y)}{\rho(y')}\right)^k} - \frac{1}{\left(1+\frac{d(x_0,y)}{\rho(y')}\right)^k} \right| \leq C \frac{k}{\rho(y')} \frac{d(x,x_0)}{\left(1+\frac{d(x_0,y)}{\rho(y')}\right)^{k+1}}$$
$$\leq C d(x_0,y)^{-1} \frac{d(x,x_0)}{\left(1+\frac{d(x_0,y)}{\rho(y')}\right)^k},$$

which we obtain from applying again the mean value Theorem.

Thus, by using the fact that $d(x_0,y) \simeq d(x,y)$, we obtain that A and B are bounded by

$$C V(y') \frac{1}{\left(1+\frac{d(x_0,y)}{\rho(y')}\right)^k} \frac{d(x,x_0)}{d(x_0,y)^{n+1}}.$$

The second step is to consider the balls $B_j = B(x_0, 2^j r)$, the annuli $C_j = \{y : 2^j r < d(y, x_0) \le 2^{j+1} r\} = \overline{B_{j+1}} \setminus \overline{B_j}$ and the rectangles $B'_j \times I_j$, where $B'_j = \{y' \in \mathbb{R}^n : |y' - x'_0| \le 2^j r\}$ and $I_j = \{s \in \mathbb{R} : |s - t_0| \le (2^j r)^2\}$. Thus, $C_j \subset B'_{j+1} \times I_{j+1}$.

In view of 2.4 replacing $\rho(y')$ with $\rho(x')$ (possibly with a change of the integer k), we have that

$$
\begin{aligned}
\Big(\int_{C_j} A^q dy\Big)^{\frac{1}{q}} &\le C \frac{1}{\big(1 + \frac{2^j r}{\rho(x')}\big)^k} \frac{r}{(2^j r)^{n+1}} \Big(\int_{C_j} V(y')^q dy\Big)^{\frac{1}{q}} \\
&\le C \frac{1}{\big(1 + \frac{2^j r}{\rho(x')}\big)^k} \frac{r}{(2^j r)^{n+1}} \Big(\int_{I_{j+1}} ds \int_{B'_{j+1}} V^q(y') dy'\Big)^{\frac{1}{q}} \\
&\le C \frac{1}{\big(1 + \frac{2^j r}{\rho(x')}\big)^k} \frac{r}{(2^j r)^{n+1}} (2^{j+1} r)^{\frac{n+2}{q}} \Big(\frac{1}{|B'_{j+1}|} \int_{B'_{j+1}} V^q(y') dy'\Big)^{\frac{1}{q}} \\
&\le C \frac{1}{\big(1 + \frac{2^j r}{\rho(x')}\big)^k} \frac{r}{(2^j r)^{n+1}} (2^j r)^{\frac{n+2}{q}} \frac{1}{(2^j r)^n} \int_{B'_{j+1}} V(y') dy',
\end{aligned}
$$

where in the last inequality we used the reverse Hölder condition on the potential V.

The third step is to add up and split, as follows:

$$
\begin{aligned}
&\sum_{j=0}^{\infty} j (2^j r)^{\frac{n+2}{q'}} \Big(\int_{C_j} A^q dy\Big)^{\frac{1}{q}} \\
&\qquad \le C \sum_{j=0}^{\infty} j (2^j r)^{n+2} \frac{1}{\big(1 + \frac{2^j r}{\rho(x')}\big)^k} \frac{r}{(2^j r)^{n+1}} \frac{1}{(2^j r)^n} \int_{B'_{j+1}} V(y') dy' \\
&\qquad \le C \sum_{j: 2^j r < \rho(x')} (\dots) + C \sum_{j: 2^j r \ge \rho(x')} (\dots) = A_I + A_{II}.
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
A_I &\le C \sum_{j: 2^j r < \rho(x')} j (2^j r)^{n+2} \frac{r}{(2^j r)^{n+1}} \frac{1}{(2^{j+1} r)^n} \int_{B'_{j+1}} V(y') dy' \\
&\le C \sum_{j: 2^j r < \rho(x')} j (2^j r)^{n+2} \frac{r}{(2^j r)^{n+1}} \Big(\frac{\rho(x')}{2^j r}\Big)^{\frac{n}{q}} \frac{1}{\rho(x')^n} \int_{B(x', \rho(x'))} V(y') dy',
\end{aligned}
$$

because of equation 2.5. Finally, by definition of ρ (see 2.2) and since $q > \frac{n}{2}$ we conclude that A_I is finite:

$$
A_I \le C \sum_{j: 2^j r < \rho(x')} \frac{j}{2^j} \Big(\frac{\rho(x')}{2^j r}\Big)^{\frac{n}{q} - 2} \le C \sum_{j: 2^j r < \rho(x')} \frac{j}{2^j}.
$$

Similarly, by using the doubling property of the measure $V(y')dy'$, equation 2.5 and definition of ρ, we have that

$$\begin{aligned}
A_{II} &\leq C \sum_{j:2^j r \geq \rho(x')} \frac{j}{2^j}(2^j r)^2 \Big(\frac{\rho(x')}{2^j r}\Big)^k \frac{1}{(2^j r)^n} \int_{B'_{j+1}} V(y')dy' \\
&\leq C \sum_{j:2^j r \geq \rho(x')} \frac{j}{2^j}(2^j r)^2 \Big(\frac{\rho(x')}{2^j r}\Big)^k \frac{1}{(2^j r)^n} \Big(\frac{2^j r}{\rho(x')}\Big)^\alpha \int_{B(x',\rho(x'))} V(y')dy' \\
&\leq C \sum_{j:2^j r \geq \rho(x')} \frac{j}{2^j} \Big(\frac{\rho(x')}{2^j r}\Big)^{k-\alpha+n-2},
\end{aligned}$$

which is finite for k large enough, and the proof of the Lemma follows. □

Lemma 2.3. *Let (X,d) be a metric space with the weak homogeneity property (hence separable) and let Λ be a nonempty open proper subset of X. Let $0 < r_0 < \beta/10$. Then, there exist two families of balls, denoted by $\mathcal{G}_r, \tilde{\mathcal{G}}_r$, such that*

$$\mathcal{W}_{r_0} = \mathcal{G}_{r_0} \cup \tilde{\mathcal{G}}_{r_0} = \{B_i\}$$

is a covering of Λ by balls of $\mathcal{F}_\beta$ with the following properties:

1. *If $B = B(x_B, s_B) \in \tilde{\mathcal{G}}_{r_0}$, then $10B \in \mathcal{F}_\beta$, $d(x_B, \Lambda^C) \leq 1$ and $\frac{1}{2} r_0 d(x_B, \Lambda^C) \leq s_B \leq r_0 d(x_B, \Lambda^C)$.*
2. *If $B \in \mathcal{G}_{r_0}$, then $B \equiv B(x_B, r_0)$, $10B \in \mathcal{F}_\beta$ and $d(x_B, \Lambda^C) > 1$.*
3. *If $B, B' \in \mathcal{W}_{r_0}$ and $B \cap B' \neq \emptyset$, then: $B \subset 5B'$ and $B' \subset 5B$.*
4. *There exists $M > 0$ such that $\sum_{B \in \mathcal{W}_{r_0}} \chi_B(x) \leq M$.*

Proof. Let $r_0 < \beta/10$ and define

$$\Lambda_k = \{x \in \Lambda : 2^{-k} < d(x, \Lambda^C) \leq 2^{-k+1}\}$$

for $k > 0$, and

$$\Lambda_0 = \{x \in \Lambda : 1 < d(x, \Lambda^C))\}.$$

We have that $\Lambda = \bigcup_{i=0}^\infty \Lambda_k$. For each $k \geq 0$ let us choose a maximal family of points $\{x_{ik}\}_{i=1}^\infty$ in Λ_k such that $d(x_{ik}, x_{ij}) > r_0 2^{-k}$. For each $k \geq 0$ let us consider the family of balls $\{B(x_{ik}, r_0 2^{-k})\}$. This family clearly verifies that $\Lambda_k \subset \bigcup_{i=1}^\infty B(x_{ik}, r_0 2^{-k})$, and

$$\Lambda = \bigcup_{k=0}^\infty \bigcup_{i=1}^\infty B(x_{ik}, r_0 2^{-k}).$$

Let us consider for each $k \geq 1$ a ball $B_{ik} = B(x_{ik}, r_{B_{ik}})$ such that $r_{B_{i_k}} = r_0 2^{-k}$. We can easily see that $\{B_{ik}\}$ is a covering of $\Lambda \setminus \Lambda_0$ such that $10B_{ik} \in \mathcal{F}_\beta$ and

$$\frac{1}{2} r_0 d(x_{ik}, \Lambda^C) < r_{B_{ik}} \leq r_0 d(x_{ik}, \Lambda^C).$$

For $k = 0$ let us consider the family $\{B_{i0}\} = \{B(x_{i0}, r_0)\}_{i=1}^\infty$. We have that $B_{i0} \in \mathcal{F}_\beta$ and $10B_{i0} \in \mathcal{F}_\beta$. If $B_{ik} \cap B_{jl} \neq \emptyset$,with $k, l \geq 0$, then:

$$B_{jl} \subset 5B_{ik}.$$

Indeed, if $z \in B_{ik} \cap B_{jl}$, then

$$2^{-k} \leq d(x_{ik}, \Lambda^C) \leq d(x_{jl}, \Lambda^C) + d(x_{jl}, z) + d(z, x_{ik}) \leq 2^{-l+1} + r_0 2^{-l} + r_0 2^k,$$

from where $2^{-k+l} \leq \frac{2+r_0}{1-r_0} < 3$, and by simmetry, also $2^{-l+k} < 3$, which leads us to $|k-l| \leq 1$. The worst possible situation is $k = l+1$. Let us consider $y \in B_{jl}$, then

$$d(y, x_{ik}) \leq d(y, x_{jl}) + d(x_{jl}, z) + d(z, x_{ik}) < r_0 2^{-l} + r_0 2^{-l} + r_0^{-k} = 5r_{ik}.$$

Thus, from the above computations, we can conclude that property 3 holds and x_{jl} is in the same band Λ_k or in a neighbour band Λ_j. Hence, the sets $\{x_{jl} \in \Lambda_j : B_{ik} \cap B_{jl} \neq \emptyset\}$, with $|k-j| \leq 1$, have at most finite cardinal which does not depend on B_{ik}. Then, there exists M, independent of r_0 and β, such that

$$\sum_{k=0}^{\infty} \sum_{i=1}^{\infty} \chi_{B_{ik}}(x) \leq M.$$

□

Let us state the following Lemma, which is often used in the paper without mentioning it.

Lemma 2.4. *Let (X, d) be a metric space and let Λ be a nonempty open proper subset of X. Let $0 < \beta < 1$ and $\alpha > 1$. Given $B_0 = B(z_0, r_0)$ such that $\alpha B_0 \in \mathcal{F}_\beta$ and any $x \in B_0$ we have that $r_0 < \frac{\beta}{\alpha-\beta} d(x, \Lambda^C)$ and $B\big(x, (\alpha-\beta)r_0\big) \in \mathcal{F}_\beta$.*

Proof. Since $\alpha B_0 \in \mathcal{F}_\beta$, we have that

$$r_0 < \frac{\beta}{\alpha} d\big(z_0, \Lambda^C\big) < \frac{\beta}{\alpha}(d(x, z_0) + d(x, \Lambda^C)) < \frac{\beta}{\alpha} r_0 + \frac{\beta}{\alpha} d(x, \Lambda^C),$$

therefore $\big(1 - \frac{\beta}{\alpha}\big) r_0 < \frac{\beta}{\alpha} d(x, \Lambda^C)$, and finally

$$(\alpha - \beta) r_0 < \beta d(x, \Lambda^C).$$

□

We also need the following version of the Fefferman-Stein inequality on spaces of homogeneous type:

Lemma 2.5 (See [13])**.** *Let (X, d, μ) be a space of homogeneous type regular in measure, such that $\mu(X) < \infty$. Let f be a positive function in L^∞ with bounded support and $w \in A_\infty$. Then, for every p, $1 < p < \infty$, there exists a positive constant $C = C([w]_{A_\infty})$ such that if $\|M_X f\|_{L^p(w)} < +\infty$, then*

$$\|M_X f\|^p_{L^p(w)} \leq C \|M^\sharp_X f\|^p_{L^p(w)},$$

where

$$M_X f(x) = \sup_{x \in P \in F(X)} \frac{1}{\mu(P \cap X|)} \int_{P \cap X} |f(y)| d\mu(y),$$

$$M^\sharp_X f(x) = \sup_{x \in P \in F(B)} \frac{1}{\mu(P \cap X)} \int_{P \cap X} |f(y) - f_{P \cap X}| d\mu(y) + \frac{1}{\mu(X)} \int_X f(y) d\mu(y),$$

with

$$F(B) = \{B(x_B, r_B) : x_B \in X, r_B > 0\}.$$

3. Previous results for the proof of the Theorem 1.1

In order to prove Theorem 1.1 we will need the following results:

Theorem 3.1 (See [4] and [13]). *Under assumptions (1) and (2), for any $p \in (1, \infty)$ and $w \in A_{p,loc}(\Omega)$, there exist C and $r_0 > 0$ such that for any ball $B_0 = B(x_0, r_0)$ in Ω with $10B_0 \in \mathcal{F}_\beta$ and any $u \in W_0^{2,p}(B_0)$ the following inequality holds*

$$\|D^2 u\|_{L^p_w(B_0)} \leq C\|Au\|_{L^p_w(B_0)}.$$

Proof. The proof follows the same lines of the proof of Lemma 4.1 in [4], which makes use of expansion into spherical harmonics on the unit sphere in $\mathbb{R}^n$. After that, all is reduced to obtain L^p- boundedness ofón-Zygmund operator T and its conmutator on a ball B contained in Ω (see Theorems 2.10, 2.11 and the representation formula (3.1) in this paper). We can look at the operator T and its conmutator $[T, b]$ acting on functions defined over the space of homogeneous type B equipped with the Euclidean metric and the restriction of Lebesgue measure. Also, it is easy to check that the weight $w\chi_B$ is in $A_p(B)$, provided w belongs to $A_{p,loc}(\Omega)$, since B has been chosen such that $10B \in \mathcal{F}_\beta$. By the weighted theory of singular integrals and conmutators on spaces of homogeneous type (see for instance [13]), applied to our operator the result follows. □

Theorem 3.2 (See [8], Proposition 4.1). *Let $1 < p < \infty$ and $w \in A_{p,loc}(\Omega)$. For any function $u \in W^{k,p}_{\delta,w}(\Omega)$, and any j, $1 \leq j \leq k-1$, and γ such that $|\gamma| = j$, we have*

$$\|\delta^j D^\gamma u\|_{L^p_w(\Omega)} \leq C(\epsilon^{-j}\|u\|_{L^p_w(\Omega)} + \epsilon^{k-j}\|\delta^k D^k u\|_{L^p_w(\Omega)}),$$

for any $0 < \epsilon < 1$ and C independent of u and ϵ, with $\delta(x) = \min\{1, d(x, \Omega)^C\}$.

The main Theorem of this section is the following:

Theorem 3.3. *Let $a_{ij} \in VMO$, for $i, j = 1, \ldots, n$, $V \in RH_q$ with $1 < p \leq q$, and $w \in A_{\frac{q-1}{q-p}p,loc}$. Then there exist positive constants C and r_0 such that for any ball $B_0 = B(z_0, r_0)$ in Ω with $10B_0 \in \mathcal{F}_\beta$ and any $u \in C_0^\infty(B_0)$, we have that*

$$\|Vu\|_{L^p_w(B_0)} \leq C\|Lu\|_{L^p_w(B_0)}.$$

Proof. For $z_0 \in \Omega$ pick a ball $B_0 := B(z_0, r_0)$ with r_0 to be chosen later. We follow the argument from [2]: let $x_0 \in B_0$ and fix the coefficients of A at x_0, namely $a_{ij}(x_0)$, to obtain the operator

$$L_0 u = -\sum_{i,j=1}^n a_{ij}(x_0) u_{x_i x_j} + Vu = A_0 u + Vu.$$

Rewrite the operator L_0 in divergence form:

$$L_0 u = -\left(\sum_{i,j=1}^n a_{ij}(x_0) u_{x_i}\right)_{x_j} + Vu.$$

From proposition 4.9 of [6] we know that the operator L_0 has a fundamental solution $\Gamma(x_0; x, y)$ which satisfies that for every positive integer k there exists a constant C_k, independent of x_0, such that

$$\Gamma(x_0; x, y) \leq C_k \frac{1}{\left(1 + \frac{|x-y|}{\rho(x)}\right)^k} \frac{1}{|x-y|^{n-2}}, \tag{3.1}$$

where $\rho(x)$ is the critical radius (recall section 2.1).

Thus, for any $u \in C_0^\infty(B_0)$, $x \in B_0$,

$$\begin{aligned} u(x) &= \int \Gamma(x_0; x, y) L_0 u(y) dy = \\ &= \int \Gamma(x_0; x, y) L u(y) dy + \int \Gamma(x_0; x, y)[A_0 u(y) - A u(y)] dy. \end{aligned}$$

Now if we let $x_0 = x$, we obtain

$$u(x) = \int \Gamma(x; x, y) L u(y) dy + \sum_{i,j=1}^{n} \int \Gamma(x; x, y)[a_{ij}(y) - a_{ij}(x)] u_{x_i x_j}(y) dy. \tag{3.2}$$

Then the following pointwise bound holds for all $k \in \mathbb{N}$, $x \in B_0$,

$$\begin{aligned} |V(x)u(x)| \le C_k V(x) \int_{B_0} \frac{1}{\left(1 + \frac{|x-y|}{\rho(x)}\right)^k} \frac{1}{|x-y|^{n-2}} \Big(|Lu(y)| + \\ + \sum_{i,j=1}^{n} |a_{ij}(y) - a_{ij}(x)| |u_{x_i x_j}(y)| \Big) dy. \end{aligned} \tag{3.3}$$

Next let us rewrite (3.3) as

$$|V(x)u(x)| \le C_k S_k(|Lu|)(x) + \sum_{i,j=1}^{n} S_{k,a_{ij}}(|u_{x_i x_j}|)(x), \tag{3.4}$$

where S_k and $S_{k,a}$ are the integral operators defined as

$$S_k f(x) = V(x) \int \frac{1}{\left(1 + \frac{|x-y|}{\rho(x)}\right)^k} \frac{1}{|x-y|^{n-2}} f(y) dy, \tag{3.5}$$

and

$$S_{k,a} f(x) = V(x) \int \frac{1}{\left(1 + \frac{|x-y|}{\rho(x)}\right)^k} \frac{1}{|x-y|^{n-2}} |a(y) - a(x)| f(y) dy, \tag{3.6}$$

with $a \in L^\infty \cap VMO(\mathbb{R}^n)$, $k \in \mathbb{N}$.

We will prove in Theorem 3.4 below that for all $p \in (1, q]$ and k large enough,

$$\|S_k f\|_{L^p_w(B_0)} \le C \|f\|_{L^p_w(B_0)}. \tag{3.7}$$

Also, we will prove in Theorem 3.5 below that for each $\epsilon > 0$ there exists $r_0 > 0$ depending on the VMO-modulus of the function a such that

$$\|S_{k,a} f\|_{L^p_w(B_0)} \le \epsilon \|f\|_{L^p_w(B_0)}. \tag{3.8}$$

Then, by (3.4), (3.7), (3.8) and Theorem 3.1 we have that for any $u \in C_0^\infty(B_0)$ with r_0 small enough,

$$\begin{aligned} \|Vu\|_{L^p_w(B_0)} &\le C \|Lu\|_{L^p_w(B_0)} + \epsilon \|u_{x_i x_j}\|_{L^p_w(B_0)} \le C \|Lu\|_{L^p_w(B_0)} + C\epsilon \|Au\|_{L^p_w(B_0)} \\ &\le (C + C\epsilon) \|Lu\|_{L^p_w(B_0)} + C\epsilon \|Vu\|_{L^p_w(B_0)}, \end{aligned}$$

and Theorem 3.3 follows.

$\square$

3.1. **Statement and proof of Theorems 3.4 and 3.5:** Following the lines of [2], let us also consider the operators defined in Ω

$$S_k^* f(x) = \int \frac{V(y)}{\left(1+\frac{|x-y|}{\rho(y)}\right)^k} \frac{1}{|x-y|^{n-2}} f(y)dy, \qquad x \in \Omega. \qquad \text{and}$$
$$S_{k,a}^* f(x) = \int \frac{V(y)}{\left(1+\frac{|x-y|}{\rho(y)}\right)^k} \frac{1}{|x-y|^{n-2}} |a(y)-a(x)| f(y)dy,$$

for each positive integer k and $a \in VMO$. These operators are the adjoint of the integral operator S_k and $S_{k,a}$, given in (3.5) and (3.6) respectively.

Theorem 3.4. *Let B_0 be a ball in $\mathcal{F}_\beta$ such that $10B_0 \in \mathcal{F}_\beta$. Then for k large enough and $p \in [1,q]$, the operator S_k is bounded on $L^p_w(B_0)$, with $w \in A_{\frac{q-1}{q-p}p,loc}(\Omega)$.*

Proof. It is enough to prove that the adjoint operator S_k^* is bounded on $L_v^{p'}(B_{r_0})$, with $v = w^{-1/p-1} \in A_{p'/q',\mathrm{loc}}(\Omega)$ for $p' \in [q', \infty]$, since $\frac{p'}{q'}$ and $\frac{q-1}{q-p}p$ are conjugate exponents. As we pointed out in section 2.1.4, we may replace $\rho(y)$ by $\rho(x)$ in the kernel of the operator S_k^* (and maybe changing the integer k). Assume, without loss of generality, that $f \geq 0$. Also assume that $q > \frac{n}{2}$, which can be done because of the fact that if V satisfies the RH_q property, then V satisfies the $RH_{q+\epsilon}$ property for some $\epsilon > 0$.

We will prove the pointwise bound

$$S_k^* f(x) \leq C(M_{\beta,\mathrm{loc}}(|f|^{q'})(x))^{\frac{1}{q'}} =: M_{q',\mathrm{loc}},$$

for $x \in B_0$, $f \in L^p_w(B_0)$ and $f \geq 0$. If $p > q'$ the theorem then follows by the boundedness of the local-maximal function (Theorem 2.1), and if $p = q'$ the theorem follows from the fact that V satisfies the $RH_{q+\epsilon}$ property for some $\epsilon > 0$.

We have that

$$\begin{aligned} S_k^* f(x) \leq C &\int_{|x-y|<\rho(x)} \frac{V(y)}{\left(1+\frac{|x-y|}{\rho(x)}\right)^k} \frac{1}{|x-y|^{n-2}} \chi_{B_0}(y) f(y)dy + \\ &+ C \int_{|x-y|\geq\rho(x)} \frac{V(y)}{\left(1+\frac{|x-y|}{\rho(x)}\right)^k} \frac{1}{|x-y|^{n-2}} \chi_{B_0}(y) f(y)dy \\ \leq C &\int_{|x-y|<\rho(x)} \frac{V(y)}{|x-y|^{n-2}} \chi_{B_0}(y) f(y)dy + \\ &+ C \int_{|x-y|\geq\rho(x)} \left(\frac{\rho(x)}{|x-y|}\right)^k \frac{V(y)}{|x-y|^{n-2}} \chi_{B_0}(y) f(y)dy = \mathbf{A}(x) + \mathbf{B}(x). \end{aligned}$$

Let $x \in B_0 = B(z_0, r_0)$.

Let us first study $\mathbf{A}(x)$. Denote by B_j the balls $B_j = B(x, 2^{-j}\rho(x))$ and by C_j the annuli defined as $C_j = \{y : 2^{-(j+1)}\rho(x) < |x-y| \leq 2^{-j}\rho(x)\} = \overline{B_j} \backslash \overline{B_{j+1}}$, $j \in \mathbb{N}_0$.

If $\rho(x) \le r_0$ then, by the Lemma 2.4 we have that $\rho(x) \le r_0 < \frac{\beta}{10-\beta} d(x, \Omega^C)$. Then $B(x, \rho(x)) \in \mathcal{F}_\beta$ and we proceed as in [2]. That is,

$$\begin{aligned}
\mathbf{A}(x) &\le C \sum_{j=0}^{\infty} \frac{1}{(2^{-j}\rho(x))^{n-2}} \int_{C_j} V(y) f(y) dy \le \\
&\le C \sum_{j=0}^{\infty} (2^{-j}\rho(x))^2 \left(\frac{1}{|B_j|} \int_{B_j} V(y)^q dy \right)^{\frac{1}{q}} \left(\frac{1}{|B_j|} \int_{B_j} f(y)^{q'} dy \right)^{\frac{1}{q'}} \\
&\le C M_{q',\mathrm{loc}}(f)(x) \sum_{j=0}^{\infty} (2^{-j}\rho(x))^2 \left(\frac{1}{|B_j|} \int_{B_j} V(y) dy \right),
\end{aligned}$$

by Hölder inequality, RH_q condition and the definition of local Maximal function of exponent q'.

A slight modification of the argument is needed in the case $\rho(x) > r_0$: there exists $j_0 \in \mathbb{N}_0$ such that $2^{-(j_0+1)}\rho(x) < r_0 \le 2^{-j_0}\rho(x)$. Let $y \in C_j$, for $j \le j_0 - 2$. Then,

$$2^{-(j+1)}\rho(x) < |x - y| \le 2^{-j}\rho(x),$$

and also

$$2r_0 < 2^{-j_0+1}\rho(x) \le 2^{-(j+1)}\rho(x),$$

from where

$$2r_0 \le 2^{-(j+1)}\rho(x) < |x - y| \le |x - z_0| + |z_0 - y| < r_0 + |z_0 - y|.$$

Therefore $|z_0 - y| > r_0$, and thus $B_0 \cap C_j = \emptyset$ if $j \le j_0 - 2$. Then,

$$\begin{aligned}
\mathbf{A}(x) &\le C \sum_{j=j_0-1}^{\infty} \frac{1}{(2^{-j}\rho(x))^{n-2}} \int_{B_0 \cap C_j} V(y) f(y)\, dy \\
&\le C \sum_{j=j_0-1}^{\infty} (2^{-j}\rho(x))^2 \big(\frac{1}{|B_j|} \int_{B_j} V(y)^q dy \big)^{\frac{1}{q}} \left(\frac{1}{|B_j|} \int_{B_j} f(y)^{q'}\, dy \right)^{\frac{1}{q'}},
\end{aligned}$$

by Hölder inequality and the fact that $C_j \subset \overline{B_j}$. Since $B_j = B(x, 2^{-j}\rho(x)) \subset B(x, 4r_0) \subset B(z_0, 5r_0)$ and $B(z_0, 10r_0) \in \mathcal{F}_\beta$, we have that $B_j \in \mathcal{F}_\beta$, $j \ge j_0 - 1$, in view of Lemma 2.4.

Then, applying the RH_q condition on V, we obtain

$$\mathbf{A}(x) \le C M_{q',\mathrm{loc}}(f)(x) \sum_{j=j_0-1}^{\infty} (2^{-j}\rho(x))^2 \left(\frac{1}{|B_j|} \int_{B_j} V(y) dy \right).$$

Finally, we follow the same steps as in [2] to conclude that

$$\mathbf{A}(x) \le C M_{q',\mathrm{loc}}(f)(x),$$

namely, choose $R = \rho(x)$ and $r = 2^{-j}\rho(x)$ in 2.5, and use 2.3 from section 2.1.3, when needed.

Next we study $\mathbf{B}(x)$.

This time, if $\rho(x) > 2r_0$ we have that $\mathbf{B}(x) = 0$.

The other case goes as follows: now consider the balls $B_j = B(x, 2^j\rho(x))$ and the annuli $C_j = \{y : 2^{j-1}\rho(x) < |x - y| \le 2^j\rho(x)\} \subset \overline{B_j} \backslash \overline{B_{j-1}}$, for $j \in \mathbb{N}_0$. There exists $j_0 \in \mathbb{N}_0$ such that $2^{j_0-1}\rho(x) < r_0 \le 2^{j_0}\rho(x)$. Consider $y \in C_j$ for $j \ge j_0 + 2$. Then,

$$2^{j-1}\rho(x) < |x - y| \le 2^j\rho(x),$$

and since $2r_0 \leq 2^{j_0+1}\rho(x) \leq 2^{j-1}\rho(x)$, we have that

$$2r_0 < |x-y| \leq |x-z_0| + |z_0-y| < r_0 + |z_0-y|.$$

Therefore, $|z_0 - y| > r_0$ and we conclude that $B_0 \cap C_j = \emptyset$, for $j \geq j_0 + 2$. Then,

$$\begin{aligned}\mathbf{B}(x) &\leq C \sum_{j=0}^{j_0+1} \frac{2^{-jk}}{(2^j\rho(x))^{n-2}} \int_{B_0\cap C_j} V(y)f(y)dy \\ &\leq C \sum_{j=0}^{j_0+1} \frac{(2^j\rho(x))^2}{2^{jk}} \Big(\frac{1}{|B_j|}\int_{B_j} V(y)^q dy\Big)^{\frac{1}{q}} \Big(\frac{1}{|B_j|}\int_{B_j} f(y)^{q'} dy\Big)^{\frac{1}{q'}},\end{aligned}$$

by Hölder inequality and the fact that $C_j \subset B_j$. Then, for $0 \leq j \leq j_0 + 1$, we have that $B(x, 2^j\rho(x)) \subset B(x, 4r_0) \subset B(z_0, 5r_0)$. Again, since $B(z_0, 10r_0) \in \mathcal{F}_\beta$, we get $B_j \in \mathcal{F}_\beta$. Thus, from the RH_q condition

$$\mathbf{B}(x) \leq CM_{q',\mathrm{loc}}(f)(x) \sum_{j=0}^{j_0+1} \frac{(2^j\rho(x))^2}{2^{jk}} \Big(\frac{1}{|B_j|}\int_{B_j} V(y)dy\Big).$$

Now we continue the proof given in [2], that is, use again 2.5 and 2.3, to conclude that

$$\mathbf{B}(x) \leq CM_{q',\mathrm{loc}}(f)(x).$$

□

Theorem 3.5. *Let $p \in (1, q]$ and $w \in A_{\frac{q-1}{q-p}p,loc}(\Omega)$. Then, given $\epsilon > 0$ there exist $r_0 > 0$, depending on the $VMO-$modulus of a, such that for any ball $B_0 = B(z_0, r_0)$ in Ω with $10B_0 \in \mathcal{F}_\beta$, the inequality*

$$\|S_{k,a}f\|_{L^p_w(B_0)} \leq \epsilon\|f\|_{L^p_w(B_0)}$$

holds for all $f \in L^p_w(B_0)$ and k large enough.

Now we can write

$$S^*_{k,a}f(x) = \int |a(y) - a(x)|W(x,y)f(y)dy,$$

where $W(x, y)$ is the kernel given in Lemma 2.2 which satisfies the $H_1(q)$ condition, and we deduce Theorem 3.5, from the following abstract result:

Theorem 3.6. *Let $w \in A_{p/q',loc}(\Lambda)$ with $q' < p < \infty$ and $\Lambda = \Omega$ or Ω_T. Let B_0 be a ball in Λ such that $10B_0 \in \mathcal{F}_\beta$. Assume that $W(x, y)$ is a non-negative kernel satisfying the $H_1(q)$ condition on the first variable, for some $q > 1$ such that the operator*

$$Tf(x) = \int W(x,y)f(y)dy$$

is bounded on $L^p_w(B_0)$. Then for $b \in BMO(\mathbb{R}^n)$ or $BMO(\mathbb{R}^{n+1})$ the operator"positive commutator"

$$T_bf(x) = \int_{B_0} |b(x) - b(y)|W(x,y)f(y)dy$$

is bounded on $L^p_w(B_0)$, and

$$\|T_bf\|_{L^p_w(B_0)} \leq C\|b\|_{BMO}\|f\|_{L^p_w(B_0)}.$$

Proof. In view of Lemma 2.5, we will prove the following pointwise inequality: for $s > q'$ there exists a constant $C > 0$ independent of b and f such that

$$M^{\sharp}_{B_0}(T_b f)(x) \le C\|b\|_{BMO}[M_{s,\text{loc}}(Tf)(x) + M_{s,\text{loc}}(f)(x)], \tag{3.9}$$

for all $x \in B_0$, where

$$M^{\sharp}_{B_0} f(x) = \sup_{x\in B, x_B\in B_0} \inf_{c>0} \frac{1}{|B\cap B_0|}\int_{B\cap B_0} |f(y) - c|dy + \frac{1}{|B_0|}\int_{B_0} |f(y)|dy.$$

Fixed $x \in B_0$ and choose $B = B(x_B, r_B)$ with $x \in B$ and $x_B \in B_0$. Thus $|B| \simeq |B \cap B_0|$. Let $\widetilde{B} = 2B = B(x_B, 2r_B)$. From Lemma 2.4 it follows that $\widetilde{B} \in \mathcal{F}_\beta$. Now for a positive function f let us split it into the sum $f = f_1 + f_2$, where $f_1 = f\chi_{\widetilde{B}}$ and $f_2 = f\chi_{\widetilde{B}^C}$.

Proceeding as in [2], we obtain the expression

$$\begin{aligned}|T_b f(y) - C_B| &\le |b(y) - b_B|\,Tf(y) + T(|b - b_B|f_1)(y)\\ &\qquad + \int_{B_0} |W(y,z) - W(x_B,z)||b(z) - b_B|f_2(z)dz\\ &= \mathbf{A}(y) + \mathbf{B}(y) + \mathbf{C}(y)\end{aligned}$$

for any $y \in B$, where $c_B = T(|b - b_B|f_2)(x_B) = \int_{B_0} |b(z) - b_B|W(x_B, z)f_2(y)dz$.

Let us first bound $\mathbf{A}(y)$. Taking average over $B \cap B_0$, for $s > q'$,

$$\begin{aligned}Av(\mathbf{A}) &= \frac{1}{|B\cap B_0|}\int_{B\cap B_0} |b(y) - b_B|Tf(y)dy\\ &\le C\Big(\frac{1}{|B|}\int_B |b(y) - b_B|^{s'}dy\Big)^{\frac{1}{s'}}\Big(\frac{1}{|B|}\int_B \chi_{B_0}|Tf(y)|^s dy\Big)^{\frac{1}{s}} \le\\ &\le C\|b\|_{BMO} M_{s,\text{loc}}(\chi_{B_0}Tf)(x),\end{aligned}$$

Choose now γ such that $s > \gamma > q'$. The computations for the average of $\mathbf{B}$ from [2] also hold in our case:

$$\begin{aligned}Av(\mathbf{B}) &\le \frac{C}{|B|}\int_B \chi_{B_0}T(|b - b_B|f_1)(x)dx \le C\Big(\frac{1}{|B|}\int_B T(|b - b_B|f_1)^\gamma(x)dx\Big)^{\frac{1}{\gamma}} \le\\ &\le C\Big(\frac{1}{|B|}\int_{\widetilde{B}} |b(x) - b_B|^\gamma |f_1(x)|^\gamma dx\Big)^{\frac{1}{\gamma}},\end{aligned}$$

since T is bounded on $L^p(\mathbb{R}^n)$(see Theorem 3.1 in [15] and Theorem 5 in [2]). Then, by Hölder's inequality,

$$\begin{aligned}Av(\mathbf{B}) &\le C\Big(\frac{1}{|\widetilde{B}|}\int_{\widetilde{B}} |f(x)|^s dx\Big)^{\frac{1}{s}}\Big(\frac{1}{|B|}\int_{\widetilde{B}} |b(x) - b_B|^{\gamma(\frac{s}{\gamma})'}\Big)^{\frac{1}{\gamma(\frac{s}{\gamma})'}} \le\\ &\le C\Big(\frac{1}{|\widetilde{B}|}\int_{\widetilde{B}} |f(x)|^s dx\Big)^{\frac{1}{s}}\Big[\Big(\frac{1}{|\widetilde{B}|}\int_{\widetilde{B}} |b(x) - b_{\widetilde{B}}|^{\gamma(\frac{s}{\gamma})'}\Big)^{\frac{1}{\gamma(\frac{s}{\gamma})'}} + |b_B - b_{\widetilde{B}}|\Big] \le\\ &\le C\|b\|_{BMO} M_{s,\text{loc}}(f)(x),\end{aligned}$$

because $|b_B - b_{\widetilde{B}}| \le C\|b\|_{BMO}$ and the John-Nirenberg inequality.

Next we choose γ such that $\frac{1}{\gamma} + \frac{1}{q} + \frac{1}{s} = 1$, and we define the balls $B_j = B(x_B, 2^j r)$ and the annuli $C_j = \{z : 2^{j-1}r < |x_B - z| \le 2^j r\}$. Like in the proof of theorem 3.4,

there exists $j_0 \in \mathbb{N}_0$ such that $C_{j_0} \cap B_0 \neq \emptyset$ and $C_{j_0+1} \cap B_0 = \emptyset$, then by Lemma 2.4, we have that $B_j \in \mathcal{F}_\beta$ for $j \leq j_0$. Then, for any $y \in B$, we have that

$$\begin{aligned}
\mathbf{C}(y) &= \int_{\widetilde{B}^C \cap B_0} |b(z)-b_B||W(y,z)-W(x_B,z)|f(z)dz \\
&\leq \sum_{j=2}^{j_0} \int_{C_j \cap B_0} |b(z)-b_B||W(y,z)-W(x_B,z)|f(z)dz \leq \\
&\leq C \sum_{j=2}^{j_0} \left(\frac{1}{|B_j|} \int_{B_j} |b(z)-b_B|^\gamma dz \right)^{\frac{1}{\gamma}} \\
&\qquad\qquad \left(\frac{1}{|B_j|} \int_{C_j} |W(y,z)-W(x_B,z)|^q d \right)^{\frac{1}{q}} \left(\frac{1}{|B_j|} \int_{B_j} |\chi_{B_0} f(z)|^s dz \right)^{\frac{1}{s}} \leq \\
&\leq C \sum_{j=2}^{j_0} |B_j| \left[\left(\frac{1}{|B_j|} \int_{B_j} |b(z)-b_{B_j}|^\gamma dz \right)^{\frac{1}{\gamma}} + |b_B - b_{B_j}| \right] \\
&\qquad\qquad \left(\frac{1}{|B_j|} \int_{C_j} |W(y,z)-W(x_B,z)|^q dz \right)^{\frac{1}{q}} M_{s,\mathrm{loc}}(f)(x) \leq \\
&\leq C\|b\|_{BMO} M_{s,\mathrm{loc}}(f)(x) \sum_{j=2}^{\infty} (2^j r)^{\frac{n}{q'}} j \left(\int_{C_j} |W(y,z)-W(x_B,z)|^q dz \right)^{\frac{1}{q}} \leq \\
&\leq C\|b\|_{BMO} M_{s,\mathrm{loc}}(f)(x),
\end{aligned}$$

because of the $H_1(q)$ condition, the John-Nirenberg inequality and the fact that $|b_B - b_{B_j}| \leq Cj\|b\|_{BMO}$. Then putting together all the above estimates, we get

$$\sup_{\substack{x \in B \\ x_B \in B_0}} \inf_{c>0} \frac{1}{|B \cap B_0|} \int_{B \cap B_0} |f(y)-c|\, dy \leq C\|b\|_{BMO} \Big(M_{s,\mathrm{loc}}(f)(x) + M_{s,\mathrm{loc}}(Tf)(x) \Big).$$

On the other hand, proceeding as above we also have

$$\begin{aligned}
\frac{1}{|B_0|} \int_{B_0} |T_b f(y)| dy &\leq \frac{1}{|B_0|} \int_{B_0} \left(|b(y)-b_{B_0}| Tf(y) + T(|b-b_{B_0}|f)(y) \right) dy \\
&\leq C\|b\|_{BMO} \Big(M_{s,\mathrm{loc}}(\chi_{B_0} Tf)(x) + M_{s,\mathrm{loc}}(f)(x) \Big)
\end{aligned}$$

Thus we obtain 3.9, which together with Lemma 2.5 and Theorem 2.1 imply the Theorem.

□

Proof of Theorem 3.5. By duality, we prove the theorem for the adjoint operator $S^*_{k,a}$ with $v = w^{-1/p-1} \in A_{p'/q',\mathrm{loc}}(\Omega)$ for $p' \in [q', \infty)$.

Applying Theorem 3.6 to the operator $S^*_{k,a}$ for k large enough we get that if $q' < p' < \infty$,

$$\|S^*_{k,a} f\|_{L^{p'}_v(B_0)} \leq C\|a\|_{BMO} \|f\|_{L^{p'}_v(B_0)},$$

and if $p' = q'$ we use again that $V \in RH_{q+\epsilon}$.

Since $a \in VMO(\mathbb{R}^n)$, there exists a bounded uniformly continuous function ϕ in $\mathbb{R}^n$ such that

$\|a-\phi\|_{BMO} < \epsilon$. Also, for $z_0 \in \Omega$ and $r_0 > 0$ there exists a uniformly continuous function ψ such that $\psi = \phi$ in $B_0 = B(z_0, r_0)$ and

$$\|\psi\|_{BMO} \le \omega_\phi(2r_0),$$

where $\omega_\phi(2r_0)$ denote the modulus of continuity of ϕ (see [4]). Choosing r_0 small enough, for all $f \in L^p_v(B_0)$, we have

$$\begin{aligned}
\|S^*_{k,a}f\|_{L^{p'}_v(B_0)} &\le \|S^*_{k,a-\phi}f\|_{L^{p'}_v(B_0)} + \|S^*_{k,\phi}f\|_{L^{p'}_v(B_0)} \\
&= \|S^*_{k,a-\phi}f\|_{L^{p'}_v(B_0)} + \|S^*_{k,\psi}f\|_{L^{p'}_v(B_0)} \\
&\le C\|a-\phi\|_{BMO}\|f\|_{L^{p'}_v(B_0)} + C\|\psi\|_{BMO}\|f\|_{L^{p'}_v(B_0)} \\
&\le C\epsilon\|f\|_{L^{p'}_v(B_0)},
\end{aligned}$$

thus, the Theorem follows.

□

4. Previous results for the proof of the Theorem 1.2

We now present the parabolic-interpolation Theorem, which makes use of the Theorem 2.1.

Theorem 4.1. *Let $1 < p < \infty$ and $w \in A_{p,loc}(\Omega_T)$. For any function $u \in W^{k,p}_{\delta,w}(\Omega_T)$, any j, $1 \le j \le k-1$, and γ such that $|\gamma| = j$, we have that*

$$\|\delta^j D^\gamma u\|_{L^p_w(\Omega_T)} \le C(\epsilon^{-j}\|u\|_{L^p_w(\Omega_T)} + \epsilon^{k-j}\|\delta^k D^k u\|_{L^p_w(\Omega_T)}). \tag{4.1}$$

for any $0 < \epsilon < 1$ and C independent of u and ϵ with $\delta(x',t) = \min\{1, d((x',t), \Omega_T^C)\}$, where D^γ denotes the derivative with respect to the first variable.

Proof. The proof follows the same lines of the proof of Theorem 3.2 of [8] with appropriate changes. We include it for completeness. We consider the following Sobolev's integral representation (see [1]):

$$|D^\gamma v(x',s)| \le C\left(\sigma^{-n-j}\int_{B(x',\sigma)} |v(y',s)| + \int_{B(x',\sigma)} \frac{|D^k v(y',s)|}{|x'-y'|^{n-k+j}}dy'\right),$$

for any $\sigma > 0$, $(x',s) \in \mathbb{R}^n \times (0,T)$ and $v \in W^{k,1}_{\text{loc}}(\mathbb{R}^{(n+1)})$.

Let us choose a Whitney' type covering $\mathcal{W}_{r_0}$ of Ω_T with $\beta = 1/2$ and $r_0 < 1/20$. For $P = B(x_P, r_P) \in \mathcal{W}_{r_0}$, take a $\mathcal{C}_0^\infty$ function η_P such that $\text{supp}(\eta_P) \subset 4P \subset \Omega_T$, $0 \le \eta_P \le 1$, and $\eta_P \equiv 1$ on $2P$.

We apply now the above inequality to $u\eta_P$ which, by our assumptions, belongs to $W^{k,1}_{\text{loc}}(\mathbb{R}^n)$. Observe that for $(x',s) \in P$ and $\sigma \le r_P$ we have $B((x',s),\sigma) \subset 2P$ and consequently $u\eta_P$ as well as its derivatives coincide with u and its derivatives when integrated over such balls.

Therefore for $(x',s) \in P$ and $\sigma \le r_P$, we obtain the above inequality with v replaced by u, namely

$$\begin{aligned}
|D^\gamma u(x',s)| &= |D^\gamma(u\eta_P)(x',s)| \\
&\le C\sigma^{-n-j}\int_{B(x',\sigma)} |u(y',s)|dy' + C\int_{B(x',\sigma)} \frac{|D^k u(y',s)|}{|x'-y'|^{n-k+j}}dy'.
\end{aligned} \tag{4.2}$$

Moreover, as is easy to check from the properties of the covering $\mathcal{W}_{r_0}$, the balls $B(x, \sqrt{2}\sigma)$, for $x \in P$ and $\sqrt{2}\sigma \le r_P$, belong to the family $\mathcal{F}_\beta$ for $\beta = 1/2$. In fact,

for $x \in P$, since from properties 1 and 2 of Whitney's Lemma we get $10P \in \mathcal{F}_\beta$, applying the Lemma 2.4 we get

$$B(x, \sqrt{2}\sigma) \subset B(x, (10-\beta)\sqrt{2}\sigma) \subset B(x, (10-\beta)r_P) \in \mathcal{F}_\beta.$$

Let $x = (x', t) \in P$. Integrating in (4.2) over $I_\sigma(t) = (t - \sigma^2, t + \sigma^2)$ and noticing that $B(x', \sigma) \times I_\sigma(t) \subset B(x, \sqrt{2}\sigma) \in \mathcal{F}_\beta$, we get

$$\begin{aligned}
\sigma^{-2}&\int_{I_\sigma(t)} |D^\gamma u(x', s)|\, ds \\
&\leq C\sigma^{-n-2-j} \iint\limits_{B(x',\sigma)\times I_\sigma(t)} |u|(y', s)\, dy' ds \, + C\sigma^{-2} \iint\limits_{B(x',\sigma)\times I_\sigma(t)} \frac{|D^k u(y', s)|}{|x' - y'|^{n-k+j}}\, dy' ds \\
&\leq C\sigma^{-j} M_{\beta,\mathrm{loc}} u(x', t) + C\sigma^{-2} \iint\limits_{B(x',\sigma)\times I_\sigma(t)} \frac{|D^k u(y', s)|}{|x' - y'|^{n-k+j}} dy' ds
\end{aligned}$$

for all $x = (x', t) \in P$ and $\sqrt{2}\sigma \leq r_P$.

As for the second term, splitting the integral dyadically, we obtain that is bounded by

$$\sigma^{k-j} \sum_{i=0}^{\infty} 2^{i(j-k)} \frac{1}{\sigma^2 |2^{-i} B(x'\sigma)|} \int_{I_\sigma(t)} \int_{2^{-i} B(x',\sigma)} |D^k u(y', s)| dy' ds. \tag{4.3}$$

Since for $x \in P$ and $\sqrt{2}\sigma \leq r_P$ all averages involved correspond to balls in $\mathcal{F}_{1/2}$ and $j < k$, the term in (4.3) is bounded by a constant times $\sigma^{k-j} M_{\beta,\mathrm{loc}} D^k u(x)$ for all $x \in P$.

Putting together both estimates and taking $\sqrt{2}\sigma = \varepsilon r_P$, using that $r_P \simeq \delta(x)$ for $x \in P$ and denoting

$$M^2_{\mathrm{loc}} f(x', t) = \sup_{\substack{s \in I_\sigma(t) \\ \sigma \leq r_P}} \frac{1}{\sigma^2} \int_{I_\sigma(t)} |f(x', s)|\, ds,$$

we obtain

$$\begin{aligned}
|D^\gamma(u)(x', t)| &\leq C M^2_{\mathrm{loc}}(D^\gamma u)(x', t) \\
&\leq C\big((\varepsilon\delta(x))^{-j} M_{\beta,\mathrm{loc}}(u)(x) + (\varepsilon\delta(x))^{k-j} M_{\beta,\mathrm{loc}}(D^k u(x)\big)
\end{aligned} \tag{4.4}$$

for a.e. $(x', t) \in P$. Since $\mathcal{W}_{r_0}$ is a covering of Ω_T and the right hand side of (4.4) no longer depends of P, we obtain that (4.4) holds for a.e. $x = (x', t) \in \Omega_T$.

Multiplying both sides by $\delta^j(x)$ and taking the norm in $L^p_w(\Omega_T)$, we arrive to

$$\|\delta^j\, D^\gamma u\|_{L^p_w(\Omega_T)} \leq C\big(\varepsilon^{-j} \|M_{\beta,\mathrm{loc}} u\|_{L^p_w(\Omega_T)} + \varepsilon^{k-j} \|M_{\beta,\mathrm{loc}}(D^k u)\|_{L^p_{w\delta^{kp}}(\Omega_T)}\big).$$

Next, we observe that if the weight w belongs to $A_{p,\mathrm{loc}}(\Omega_T)$ also does $w\delta^s$, for any real number s. In fact, for any ball B in $\mathcal{F}_{1/2}$ we have that $\delta(x) \simeq \delta(x_B)$, for any $x \in B$ so that (1.2) holds provided it is satisfied by w.

Therefore, an application of the continuity results for $M_{\beta,\mathrm{loc}} f$, given in Theorem 2.1, leads to the interpolation inequality (4.1).

□

Next we state the parabolic version of Theorem 3.1.

Theorem 4.2 (See [3] and [13]). *Under assumptions (1) and (2), for any $p \in (1,\infty)$ and $w \in A_{p,loc}(\Omega_T)$, there exist C and $r_0 > 0$ such that for any ball $B_0 = B(z_0, r_0)$ in Ω_T with $10B_0 \in \mathcal{F}_\beta$ and any $u \in W_0^{2,p}(B_0)$ the following inequalities hold*

$$\begin{aligned} \|u_{x_ix_j}\|_{L^p_w(B_0)} &\leq C\|A_P u\|_{L^p_w(B_0)}, \\ \|u_t\|_{L^p_w(B_0)} &\leq C\|A_P u\|_{L^p_w(B_0)}. \end{aligned}$$

Proof. The proof is similar to the elliptic case, as is proved in Corollary 2.13 in [3], by using again expansion into spherical harmonics on the unit sphere, this time in $\mathbb{R}^{n+1}$. After that, all is reduced to obtain L^p- boundedness of a parabolic Calderón-Zygmund operator T and its conmutator on a ball B contained in Ω_T (see Theorems 2.12 and the representation formula (1.4) in this paper). We can look at the operator T and its conmutator $[T, b]$ acting on functions defined over the space of homogeneous type B equipped with the parabolic metric and the restriction of Lebesgue measure. As before, the weight $w\chi_B$ is in $A_p(B)$. By the weighted theory of singular integrals and conmutators on spaces of homogeneous type, (see again [13]), applied to our operators the result follows. □

Now we focus our attention in the proofs of the main Theorem of this section, that is, the parabolic version of Theorem 3.3.

Theorem 4.3. *Let $a_{ij} \in VMO(\mathbb{R}^{n+1})$, for $i,j = 1,\ldots,n$, $V \in RH_q(\mathbb{R}^n)$ with $1 < p \leq q$, and $w \in A_{\frac{q-1}{q-p}p,loc}(\Omega_T)$. Then there exist positive constants C and r_0 such that for any ball $B_0 = B(z_0, r_0)$ in Ω_T with $10B_0 \in \mathcal{F}_\beta$ and any $u \in C_0^\infty(B_0)$, we have that*

$$\|Vu\|_{L^p_w(B_0)} \leq C\big\|Lu\|_{L^p_w(B_0)}.$$

Proof. For $z_0 = (z_0', \tau) \in \Omega_T$ pick a ball $B_0 := B(z_0, r_0)$ with r_0 to be chosen later. Again we let $x_0 \in B_0$ and fix the coefficients $a_{ij}(x_0)$ to obtain the operator

$$L_0 u = u_t - \sum_{i,j=1}^n a_{ij}(x_0) u_{x_i} u_{x_j} + Vu = A_0 u + Vu.$$

From [11] we know that the fundamental solution for this operator is bounded by the expression (see section 2.1.4):

$$|\Gamma(x_0, x, y)| \leq C_k \frac{1}{\left(1 + \frac{d(x,y)}{\rho(x')}\right)^k} \frac{1}{d(x,y)^n},$$

for every $x = (x', t), y = (y', s) \in \Omega_T$, $t > s$, $k > 0$, and for some constants C_k, C_0 independent of x_0. Here again $\rho(x')$ is the critical radious.

As usual, we defreeze the coefficients to obtain 3.2 and again the following pointwise bound holds for all $k \in \mathbb{N}$, $x \in B_0$,

$$|V(x')u(x)| \leq C_k V(x') \int_{B_0} \frac{1}{\left(1 + \frac{d(x,y)}{\rho(x')}\right)^k} \frac{1}{d(x,y)^n} \Bigg(|Lu(y)| + \sum_{i,j=1}^n |a_{ij}(y) - a_{ij}(x)||u_{x_ix_j}(y)|\Bigg) dy, \tag{4.5}$$

and rewrite (4.5) as

$$|V(x')u(x)| \le C_k S_k(|Lu|)(x) + \sum_{i,j=1}^{n} S_{k,a_{ij}}(|u_{x_i x_j}|)(x), \tag{4.6}$$

where S_k and $S_{k,a}$ are the integral operators defined as

$$S_k f(x) = V(x') \int \frac{1}{\left(1+\frac{d(x,y)}{\rho(x')}\right)^k} \frac{1}{d(x,y)^n} f(y)dy, \qquad \text{and}$$
$$S_{k,a} f(x) = V(x') \int \frac{1}{\left(1+\frac{d(x,y)}{\rho(x')}\right)^k} \frac{1}{d(x,y)^n} |a(y)-a(x)| f(y)dy,$$

with $a \in L^\infty \cap VMO(\mathbb{R}^n)$, $k \in \mathbb{N}$.

Thus, as in the elliptic case, the Theorem follows from Theorem 4.2 and the next parabolic version of Theorems 3.4 and 3.5.

□

Now we need to prove the following parabolic version of Theorem 3.4:

Theorem 4.4. *Let B_0 be a ball in $\mathcal{F}_\beta$ such that $10B_0 \in \mathcal{F}_\beta$. Then for k large enough and $p \in [1,q]$, the operator S_k is bounded on $L^p_w(B_{r_0})$, with $w \in A_{\frac{q-1}{q-p}p,loc}(\Omega_T)$.*

Proof. This proof is also done by duality. The remarks we made along the proof of Theorem 3.4 also hold this time so we won't mention them.

The adjoint operator of S_k is

$$S_k^* f(x) = \int \frac{V(y')}{\left(1+\frac{d(x,y)}{\rho(y')}\right)^k} \frac{1}{d(x,y)^n} f(y)dy, \qquad x \in \Omega_T.$$

Just like before we can split

$$\begin{aligned} S_k^* f(x) \le C &\int_{d(x,y)<\rho(x')} \frac{1}{d(x,y)^n} V(y')\chi_{B_0}(y) f(y)dy + \\ &+ C \int_{d(x,y)\ge\rho(x')} \left(\frac{\rho(x')}{d(x,y)}\right)^k \frac{1}{d(x,y)^n} V(y')\chi_{B_0}(y) f(y)dy \\ &= \mathbf{A}(x) + \mathbf{B}(x). \end{aligned}$$

We will prove the pointwise bound

$$S_k^* f(x) \le C M_{q',\mathrm{loc}}(f)(x).$$

In order to study $\mathbf{A}(x)$, let $x \in B_0 = B(z_0, r_0)$. Denote by B_j the balls $B_j = B(x, 2^{-j}\rho(x'))$, by C_j the annuli defined as $C_j = \{y : 2^{-(j+1)}\rho(x') < d(x,y) \le 2^{-j}\rho(x)\} = \overline{B_j} \backslash \overline{B_{j+1}}$, and by R_j the rectangles $R_j = B'_j \times I_j$ where B'_j denotes the ball in $\mathbb{R}^n$, $B'_j = B(x', 2^{-j}\rho(x'))$ and I_j denotes the real ball $I_j = B(t, (2^{-j}\rho(x')^2)$, $j \in \mathbb{N}_0$. We have that $C_j \subset B_j \subset R_j$, and let us remark that the ball measures are $|B_j| = c_n(2^{-j}\rho(x))^{n+2}$ and $|B'_j| = C_n(2^{-j}\rho(x'))^n$. The same steps as before prove that

$$\mathbf{A}(x) \le C M_{q',\mathrm{loc}}(f)(x),$$

for $x \in B_0$, $f \in L^p_w(B_0)$ and $f \geq 0$, where $M_{q',\mathrm{loc}}$ denotes the local maximal function of exponent q', in the parabolic setting. Indeed, if $\rho(x') \leq r_0$ we have that

$$\begin{aligned}\mathbf{A}(x) &\leq C\sum_{j=0}^{\infty}\frac{|B_j|}{(2^{-j}\rho(x'))^n}\left(\frac{1}{|B'_j|}\int_{B'_j}V(y')^q dy'\right)^{\frac{1}{q}}\left(\frac{1}{|B_j|}\int_{B_j}f(y)^{q'}dy\right)^{\frac{1}{q'}}\\ &\leq CM_{q',\mathrm{loc}}(f)(x)\sum_{j=0}^{\infty}(2^{-j}\rho(x'))^2\left(\frac{1}{|B'_j|}\int_{B'_j}V(y')dy'\right),\end{aligned}$$

because of the Hölder inequality, the reverse Hölder condition V and the definition of local maximal function. And in the case $\rho(x') > r_0$, again there exists $j_0 \in \mathbb{N}_0$ such that $C_j \cap B_0 = \emptyset$ for $j \leq j_0 + 2$. The same steps as before show us that

$$\mathbf{A}(x) \leq CM_{q',\mathrm{loc}}(f)(x)\sum_{j=j_0-1}^{\infty}(2^{-j}\rho(x'))^2\left(\frac{1}{|B'_j|}\int_{B'_j}V(y')dy'\right).$$

Now we use again equations (2.5) and (2.3) to conclude that $\mathbf{A}(x) \leq CM_{q',\mathrm{loc}}(f)(x)$.

To study $\mathbf{B}(x)$, we consider the balls $B_j = B(x, 2^j\rho(x'))$, the annuli $C_j = \{y : 2^j\rho(x') < d(x,y) \leq 2^{j+1}\rho(x')\}$, and the rectangles $R_j = B'_j \times I_j = B(x', 2^j\rho(x')) \times B(t, (2^j\rho(x'))^2) \subset \mathbb{R}^n \times \mathbb{R}$, for $j \in \mathbb{N}_0$. We have that $C_j \subset B_j \subset R_j$. Observe that if $\rho(x') > 2r_0$, then $\mathbf{B}(x) = 0$, thus we consider only the case $\rho(x') \leq 2r_0$. There exists $j_0 \in \mathbb{N}_0$ such that $C_j \cap B_0 = \emptyset$ if $j \geq j_0 + 2$. Thus we have that

$$\mathbf{B}(x) \leq CM_{q',\mathrm{loc}}(f)(x)\sum_{j=0}^{j_0+1}\frac{(2^j\rho(x'))^2}{2^{jk}}\left(\frac{1}{|B'_j|}\int_{B'_j}V(y')dy'\right),$$

because of the use of Hölder inequality, the reverse Hölder conditionon V and the definition of local maximal function of the order q'. Thus, using again equations 2.5 and 2.3, $\mathbf{B}(x) \leq CM_{q',\mathrm{loc}}(f)(x)$.

□

Remark 4.5. *We note that arguing in a similar way as in the proof of Theorem 4.4 it can be show that the operator S_k is bounded on $L^p(\mathbb{R}^{(n+1)})$ with $w = 1$ and $p \in [1, q]$. In this case the operator is pointwisely bounded by the maximal Hardy-Littlewood function of order q'.*

We turn now to the proof of parabolic Theorem 3.5:

Theorem 4.6. *Let $p \in (1, q]$ and $w \in A_{\frac{q-1}{q-p}p,loc}(\Omega_T)$. Then, given $\epsilon > 0$ there exist $r_0 > 0$, depending on the $VMO-$modulus of a such that for any ball $B_0 = B(z_0, r_0)$ in Ω_T with $10B_0 \in \mathcal{F}_\beta$, the inequality*

$$\|S_{k,a}f\|_{L^p_w(B_0)} \leq \epsilon\|f\|_{L^p_w(B_0)}. \tag{4.7}$$

holds for all $f \in L^p_w(B_0)$ and k large enough.

Proof. This proof is also done by duality as in the proof of Theorem 3.5, and follows by Theorem 3.6 with $\Lambda = \Omega_T$ and $b \in BMO(\mathbb{R}^{n+1})$.

Here,

$$S^*_{k,a}f(x) = \int\frac{V(y')}{\left(1 + \frac{d(x,y)}{\rho(y')}\right)^k}\frac{1}{d(x,y)^n}|a(y) - a(x)|f(y)dy,$$

for each positive integer k and $a \in VMO$; and the kernel is

$$w(x,y) = \frac{1}{\left(1+\frac{d(x,y)}{\rho(x')}\right)^k}\frac{1}{d(x,y)^n},$$

which satisfies the $H_1(q)$ condition as shown in section 2.2 (Lemma2.2)

□

5. Proof of the Main Result

We are in position to proof Theorem 1.1.

Proof of Theorem 1.1. Let $\mathcal{W}_{r_0} = \{B_i = B(x_i, r_i)\}$ be a covering as in Lemma 2.3, with r_0 as in Theorems 3.1 and 3.3 and $0 < r_0 < \beta/10$. For each $B_i \in W_{r_0}$ we consider a function η_i such that the family $\{\eta_i\}_{i=1}^{\infty}$ satisfies

(1) $\eta_i \in \mathcal{C}_0^\infty(2B(x_i, r_i))$, $\eta_i \equiv 1$ in B_i,

(2) $\|\eta_i\|_\infty \le 1$, $\|D^\alpha \eta_i\|_\infty \le C r_i^{-|\alpha|}$ where $r_i \approx d(x_i, \partial\Omega)$ if $B(x_i, r_i) \in \tilde{\mathcal{G}}_{r_0}$ and $r_i \approx 1$ when $B(x_i, r_i) \in \mathcal{G}_{r_0}$,

(3) $\sum_{i=1}^{\infty} \chi_{2B_i}(x) \le M$.

By using Theorem 3.1, for each i, we get

$$\begin{aligned}
&\|\chi_{B_i} D^2(u\eta_i)\|^p_{L^p_w(2B_i)} \\
&\qquad \le C\|A(u\eta_i)\|^p_{L^p_w(2B_i)} \\
&\qquad \le C\big(\|Au\|^p_{L^p_w(2B_i)} + r_i^{-1}\|Du\|^p_{L^p_w(2B_i)} + r_i^{-2}\|u\|^p_{L^p_w(B_i)}\big)^p \\
&\qquad \le C\big(\|Au\|_{L^p_w(2B_i)} + r_i^{-1}\|Du\|_{L^p_w(2B_i)} + r_i^{-2}\|u\|_{L^p(2B_i)}\big)^p \\
&\qquad \le C\big(\|Lu\|_{L^p_w(2B_i)} + \|Vu\|_{L^p_w(2B_i)} + r_i^{-1}\|Du\|_{L^p_w(2B_i)} + r_i^{-2}\|u\|_{L^p_w(2B_i)}\big)^p.
\end{aligned}$$

Analogously, using this time Theorem 3.3, since $w \in A_{p,\text{loc}}(\Omega) \subset A_{\frac{q-1}{q-p}p,\text{loc}}(\Omega)$ we obtain

$$\begin{aligned}
\|\chi_{B_i} V(u\eta_i)\|^p_{L^p_w(2B_i)} &\le C\|L(u\eta_i)\|^p_{L^p_w(2B_i)} \\
&\le C\|Lu\|^p_{L^p_w(B_i)} + r_i^{-1}\|Du\|^p_{L^p_w(B_i)} + r_i^{-2}\|u\|^p_{L^p_w(B_i)}.
\end{aligned}$$

Now, we note that for $x \in B_i$ the function $\eta_i u$ coincides with u, and also for $x \in 2B_i$, we have $\delta(x_i) \approx r_i$ with $\delta(x_i) = \min\{1, d(x_i, \partial\Omega)\}$. Hence, putting together both estimates, multiplying both sides by δ^2, adding over i, using de finite overlapping property of the covering $\{2B_i\}$ and taking the $1/p$-th power, we arrive to

$$\|u\|_{W^{2,p}_{\delta,w}(\Omega)} + \|\delta^2 Vu\|_{L^p_w(\Omega)} \le C(\|\delta^2 Lu\|_{L^p_w(\Omega)} + \|\delta Du\|_{L^p_w(\Omega)} + \|u\|_{L^p_w(\Omega)}).$$

Using the interpolation Theorem 3.2

$$\le C(\|\delta^2 Lu\|_{L^p_w(\Omega)} + \epsilon\|\delta^2 D^2 u\|_{L^p_w(\Omega)}) + (C+\epsilon^{-1})\|u\|_{L^p_w(\Omega)}.$$

Finally, choosing ϵ such that $C\epsilon = 1/2$ and subtracting the term $\|\delta^2 D^2 u\|_{L^p_w(\Omega)}$, it follows

$$\|u\|_{W^{2,p}_{\delta,w}(\Omega)} \le C\{\|Lu\|_{L^p_w(\Omega)} + \|u\|_{L^p_w(\Omega)}\},$$

whence the desired estimate follows.

□

The proof of Theorem 1.2 is obtained by a few changes:

Proof of Theorem 1.2. Just like in the previous proof, from Lemma 2.3 applying this time to $\Gamma = \Omega_T$, we consider a covering $\mathcal{W}_{r_0}$ and a family $\{\eta_i\}$ which satisfies 1 and 3, and the following 2: $\|\eta_i\|_\infty \leq 1$,

$$\|D_x^\alpha \eta_i\|_\infty \leq C r_i^{-|\alpha|},$$
$$\|D_t \eta_i\|_\infty \leq C r_i^{-2},$$

where $r_i \approx d(x_i, \partial\Omega)$ if $B(x_i, r_i) \in \tilde{\mathcal{G}}_{r_0}$ and $r_i \approx 1$ when $B(x_i, r_i) \in \mathcal{G}_{r_0}$.

Now for each i we use theorems 4.2 and 4.3 to get

$$\begin{aligned}
\|\chi_{B_i} D_x^2(u\eta_i)\|_{L^p_w(2B_i)} &\leq C\big(\|Lu\|_{L^p_w(2B_i)} + \|Vu\|_{L^p_w(2B_i)} + \\
&\qquad + r_i^{-1}\|Du\|_{L^p_w(2B_i)} + r_i^{-2}\|u\|_{L^p_w(2B_i)}\big),\\
\|\chi_{B_i} D_t(u\eta_i)\|^p_{L^p_w(2B_i)} &\leq C\big(\|Lu\|_{L^p_w(2B_i)} + \|Vu\|_{L^p_w(2B_i)} + \\
&\qquad + r_i^{-1}\|Du\|_{L^p_w(2B_i)} + r_i^{-2}\|u\|_{L^p_w(2B_i)}\big),\\
\|\chi_{B_i} V u\eta_i\|^p_{L^p_w(2B_i)} &\leq C\big(\|Lu\|_{L^p_w(2B_i)} + r_i^{-1}\|D_x u\|_{L^p_w(2B_i)} + r_i^{-2}\|u\|_{L^p_w(2B_i)}\big),
\end{aligned}$$

then, by performing analogous operations to the previous Theorem, we obtain

$$\|u\|_{W^{2,p}_{\delta,w}(\Omega_T)} + \|\delta^2 Vu\|_{L^p_w(\Omega_T)} \leq C\big(\|\delta^2 Lu\|_{L^p_w(\Omega_T)} + \|\delta D_x u\|_{L^p_w(\Omega_T)} + \|u\|_{L^p_w(\Omega_T)}\big).$$

From the interpolation Theorem 4.1 we have that

$$\|\delta D_x u\|_{L^p_w(\Omega_T)} \leq C\big(\epsilon^{-1}\|u\|_{L^p_w(\Omega_T)} + \epsilon\|\delta^2 D_x^2 u\|_{L^p_w(\Omega_T)}\big),$$

which finally leads us to

$$\|u\|_{W^{2,p}_{\delta,w}(\Omega_T)} + \|\delta^2 Vu\|_{L^p_w(\Omega_T)} \leq C\big(\|\delta^2 Lu\|_{L^p_w(\Omega_T)} + \|u\|_{L^p_w(\Omega_T)}\big)$$

as we desired. $\square$

References

[1] V. I. Burenkov, "Sobolev spaces on domains", Teubner-Texte zur Mathematik, 137, Teubner, Stuttgart, 1998.

[2] M. Bramanti, L. Brandolini, E. Harboure, B. Viviani, *Global $W^{2,p}$ estimates for nondivergence elliptic operators with potentials satisfying a reverse Hölder condition*, Ann. Mat. Pura Appl. (4) **191** (2012), no. 2, 339–362.

[3] M. Bramanti and M. C. Cerutti, *$W_p^{1,2}$ solvability for the Cauchy-Dirichlet problem for parabolic equations with VMO coefficients,* Comm. Partial Differential Equations **18** (1993), no. 9-10, 1735–1763.

[4] F. Chiarenza, M. Frasca and P. Longo, *Interior $W^{2,p}$ estimates for nondivergence elliptic equations with discontinuous coefficients,* Ricerche Mat. **40** (1991), no. 1, 149–168.

[5] F. Chiarenza, M. Frasca and P. Longo, *$W^{2,p}$-solvability of the Dirichlet problem for nondivergence elliptic equations with VMO coefficients,* Trans. Amer. Math. Soc. **336** (1993), no. 2, 841–853.

[6] J. Dziubański, *Note on H^1 spaces related to degenerate Schrödinger operators,* Illinois J. Math. **49** (2005), no. 4, 1271–1297.

[7] D. Gilbarg and N. S. Trudinger, "Elliptic partial differential equations of second order", second edition, Grundlehren der Mathematischen Wissenschaften, 224, Springer, Berlin, 1983.

[8] E. Harboure, O. Salinas, B. Viviani. *Local maximal function and weights in a general setting,* Math. Ann., (2013), 1–20.

[9] C.-C. Lin and K. Stempak, *Local Hardy-Littlewood maximal operator,* Math. Ann. **348** (2010), no. 4, 797–813.

[10] C.-C. Lin, K. Stempak and Y.-S. Wang, *Local maximal operators on measure metric spaces,* Publ. Mat. **57** (2013), no. 1, 239–264.

[11] K. Kurata, *An estimate on the heat kernel of magnetic Schrödinger operators and uniformly elliptic operators with non-negative potentials,* J. London Math. Soc. (2) **62** (2000), no. 3, 885–903.

[12] A. Nowak and K. Stempak, *Weighted estimates for the Hankel transform transplantation operator,* Tohoku Math. J. (2) **58** (2006), no. 2, 277–301.

[13] G. Pradolini and O. Salinas, *Commutators of singular integrals on spaces of homogeneous type,* Czechoslovak Math. J. **57(132)** (2007), no. 1, 75–93.

[14] Z. W. Shen, *On the Neumann problem for Schrödinger operators in Lipschitz domains,* Indiana Univ. Math. J. **43** (1994), no. 1, 143–176.

[15] Z. W. Shen, *L^p estimates for Schrödinger operators with certain potentials,* Ann. Inst. Fourier (Grenoble) **45** (1995), no. 2, 513–546.

[16] S. Thangavelu, *Riesz transforms and the wave equation for the Hermite operator,* Comm. Partial Differential Equations **15** (1990), no. 8, 1199–1215.

Departamento de Matemática, Fac. de Cs. Exactas, Ingeniería y Agrimensura, Universidad Nacional de Rosario, Pellegrini 250, 2000 Rosario

E-mail address: isolda@fceia.unr.edu.ar

Facultad de Ciencias Exactas, Universidad Nacional del Centro de la Provincia de Buenos Aires, Pinto 399, 7000 Tandil

E-mail address: pviola@exa.unicen.edu.ar

Instituto de Matemática Aplicada del Litoral, CONICET- Universidad Nacional del Litoral, IMAL-CCT CONICET Santa FE, Colectora Ruta Nac. No 168, Paraje El Pozo 3000 Santa Fe

E-mail address: vivani@santafe-conicet.gov.ar